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1987 J. Phys. A: Math. Gen. 20 5737

(http://iopscience.iop.org/0305-4470/20/16/049)

J. Phys. A: Math. Gen. **20** (1987) 5737–5740. Printed in the UK

COMMENT

Ising model on the union jack lattice as a free fermion model

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Received 18 May 1987

Abstract. It is shown that the Ising model on the union jack lattice with general anisotropic and staggered interactions is equivalent to a free fermion model, and from this equivalence explicit expressions for its free energy and a sublattice spontaneous magnetisation are obtained.

The Ising model on the union jack lattice was first considered by Vaks *et al* (1965) as a system exhibiting a re-entrant transition in the presence of competing interactions. Vaks *et al* considered a union jack lattice with symmetric interactions and, after considerable effort, obtained its free energy and a sublattice two-spin correlation function. A more general version of the model of Vaks *et al* was discussed later by Sacco and Wu (1975) as a 32-vertex model on the triangular lattice. In view of the growing interest in Ising systems with competing interactions (see, e.g., Liebmann 1986), it appears useful to present here a simple formulation of the general union jack Ising model which treats the model with anisotropic and staggered interactions in a unified fashion. We show that this general union jack Ising system is equivalent to a free fermion model (Fan and Wu 1970) and from this equivalence we obtain its free energy and a sublattice spontaneous magnetisation, thus permitting a relatively simple and systematic treatment of an otherwise complicated system.

Consider the union jack lattice $\mathscr{L}_l$ of N sites, shown in figure 1, with periodic boundary conditions. Denote the Ising spin at the ith site by $\sigma_i = \pm 1$ and the six interactions $-J_1$, $-J_2$, $-J_3$, $-J_4$, $-J$, $-J'$ by $-J_r$, with r denoting the class of the edges. The partition function of the Ising model on $\mathscr{L}_l$ is thus

$$Z_l = \sum_{\sigma} \left(\sum_{\langle ij \rangle} K_r \sigma_i \sigma_j \right) \tag{1}$$

where $K_r = J_r/kT$, the first sum is over all spin configurations $\sigma_i = \pm 1$ and the second sum is over all edges $\langle ij \rangle$ of $\mathscr{L}_l$.

The lattice $\mathscr{L}_l$ consists of two sublattices $\mathscr{L}'$ and $\mathscr{L}''$, denoted by open and full circles, respectively, in figure 1. Introducing a trick used by Baxter (1986)¶ by summing over the $\frac{1}{2}N$ spins (full circles) on $\mathscr{L}''$, we can rewrite (1) as

$$Z_l = \sum_{\sigma'} \prod_{\langle i,j,k,l \rangle} \omega(\sigma_i, \sigma_j, \sigma_k, \sigma_l) \tag{2}$$

where the product is over all $\frac{1}{2}N$ faces of the sublattice $\mathscr{L}'$, each surrounded by four spins i, j, k, l as arranged in figure 1. Here, the Boltzmann factor $\omega(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$ is

‡ Work supported in part by NSF Grants DMR-8219254 and INT-8517155.
|| Work supported in part by the National Research Council.
¶ Whenever possible, we follow the notation in Baxter (1986) for easy reference.

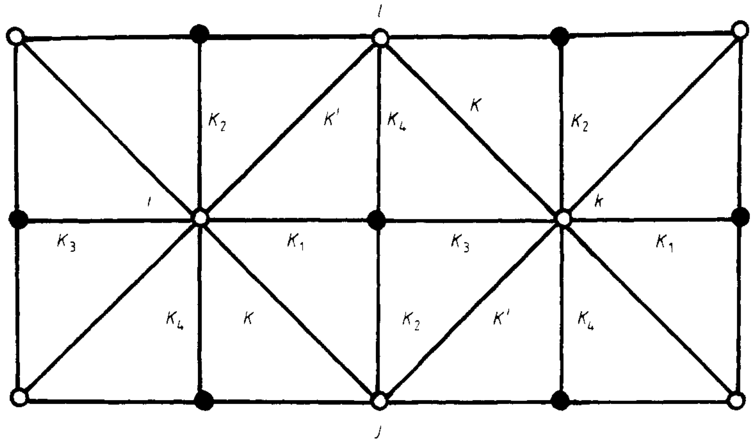

Figure 1. The union jack lattice $\mathscr{L}_l$ and the associated free fermion model lattice $\mathscr{L}'$ (open circles).

obtained by dividing each of the diagonal interactions $-J$ and $-J'$ into two halves, one for each of the two adjacent faces. This leads to

$$\begin{aligned}\omega(a, b, c, d) = 2 \exp[K(ab+cd)/2 + K'(ad+bc)/2] \\ \times \cosh(aK_1 + bK_2 + cK_3 + dK_4).\end{aligned} \tag{3}$$

Clearly, $\omega(a, b, c, d)$ is invariant under the negation of all four spin variables and (3) leads to eight distinct expressions:

$$\begin{aligned}
\omega_1 &\equiv \omega(++++) = 2\,\mathrm{e}^{K+K'} \cosh(K_1+K_2+K_3+K_4)\\
\omega_2 &\equiv \omega(+-+-) = 2\,\mathrm{e}^{-K-K'} \cosh(K_1-K_2+K_3-K_4)\\
\omega_3 &\equiv \omega(+--+) = 2\,\mathrm{e}^{-K+K'} \cosh(K_1-K_2-K_3+K_4)\\
\omega_4 &\equiv \omega(++--) = 2\,\mathrm{e}^{K-K'} \cosh(K_1+K_2-K_3-K_4)\\
\omega_5 &\equiv \omega(+-++) = 2\cosh(K_1-K_2+K_3+K_4)\\
\omega_6 &\equiv \omega(+++-) = 2\cosh(K_1+K_2+K_3-K_4)\\
\omega_7 &\equiv \omega(++-+) = 2\cosh(K_1+K_2-K_3+K_4)\\
\omega_8 &\equiv \omega(-+++) = 2\cosh(-K_1+K_2+K_3+K_4).
\end{aligned} \tag{4}$$

We now consider the dual $\mathscr{L}'_{\mathrm{D}}$ of the lattice $\mathscr{L}'$, and draw full lines on the edges of $\mathscr{L}'_{\mathrm{D}}$ if, and only if, the two adjacent spins on $\mathscr{L}'$ are opposite. Then at each site of $\mathscr{L}'_{\mathrm{D}}$ there can be eight line arrangements, as shown in figure 2. We are thus led to consider an eight-vertex model on $\mathscr{L}'_{\mathrm{D}}$. The eight-vertex model with weights (4) satisfies the free fermion condition (Fan and Wu 1970)

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8 \tag{5}$$

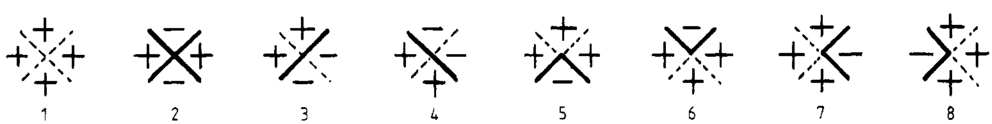

Figure 2. The eight line arrangements at a vertex of $\mathscr{L}'_{\mathrm{D}}$.

and is therefore a free fermion model with the partition function

$$Z = \tfrac{1}{2} Z_I. \tag{6}$$

The factor $\frac{1}{2}$ in (6) takes account of the two-to-one mapping of the spin and vertex configurations.

The free energy per spin of the free fermion model has been evaluated by Fan and Wu (1970). For completeness we include the expression here:

$$\begin{aligned} f &\equiv \lim_{N\to\infty} \frac{1}{N} \ln Z_I \\ &= \frac{1}{16\pi^2} \int_0^{2\pi} \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{d}\phi \ln[2a + 2b\cos\theta + 2c\cos\phi \\ &\quad + 2d\cos(\theta - \phi) + 2e\cos(\theta + \phi)] \end{aligned} \tag{7}$$

where

$$\begin{aligned} &a = \tfrac{1}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2) \qquad b = \omega_1\omega_3 - \omega_2\omega_4 \\ &c = \omega_1\omega_4 - \omega_2\omega_3 \qquad d = \omega_3\omega_4 - \omega_7\omega_8 \qquad e = \omega_3\omega_4 - \omega_5\omega_6. \end{aligned} \tag{8}$$

In writing down (7), we have used the fact that the lattice $\mathscr{L}'$ (or $\mathscr{L}'_{\mathrm{D}}$) on which the free fermion model is defined has $\frac{1}{2}N$ sites. The spontaneous magnetisation M for the free fermion model has been computed by Baxter (1986). The expression is

$$M = \begin{cases} (1 - \Omega^{-2})^{1/8} & \Omega^{-2} \leqslant 1 \\ 0 & \Omega^{-2} \geqslant 1 \end{cases} \tag{9}$$

where

$$\begin{aligned} \Omega^2 &= 1 - (-\omega_1 + \omega_2 + \omega_3 + \omega_4)(\omega_1 - \omega_2 + \omega_3 + \omega_4) \\ &\quad \times (\omega_1 + \omega_2 - \omega_3 + \omega_4)(\omega_1 + \omega_2 + \omega_3 - \omega_4)/16\omega_5\omega_6\omega_7\omega_8 \\ &= [(\omega_1^2 + \omega_2^2 - \omega_3^2 - \omega_4^2)^2 - 4(\omega_5\omega_6 - \omega_7\omega_8)^2]/16\omega_5\omega_6\omega_7\omega_8. \end{aligned} \tag{10}$$

Here we have used the free fermion condition (5) in writing down the second expression in (10).

The system exhibits an Ising transition at the critical point(s)

$$\Omega^2 = 1 \tag{11a}$$

or, equivalently,

$$\omega_1 + \omega_2 + \omega_3 + \omega_4 = 2\max\{\omega_1, \omega_2, \omega_3, \omega_4\}. \tag{11b}$$

Note that (11*b*) can also be obtained by setting the quantity inside the square brackets in (7) equal to zero at $\{\theta, \phi\} = \{0, \pm\pi, 2\pi\}$. Particularly, for $J_1 = J_2 = J_3 = J_4$ and $J = J'$, (11*b*) and (9) reduce to (7) and (20), respectively, given by Vaks *et al* (1965).

It should be pointed out that the order parameter M of the free fermion model can be obtained in two different ways. Vaks *et al* (1965) and Baxter (1986) computed the correlation function $M^2 \equiv \langle\sigma_i\sigma_j\rangle$ between two spins σ_i and σ_j (on $\mathscr{L}'$) separated by a large distance. Alternatively, by considering the free fermion model as a Z-invariant Ising model (Baxter 1978, 1986), the spontaneous magnetisation M can be obtained from the related symmetric eight-vertex model result first conjectured by Barber and Baxter (1973) on the basis of series expansions and later established by

Baxter (1982) by identifying M as $\langle\sigma_i\rangle$ for a spin σ_i deep inside $\mathscr{L}'$ with all boundary spins fixed at $\sigma=+1$. The expression for M thus obtained is precisely (9), and M is therefore a sublattice spontaneous magnetisation for the union jack lattice $\mathscr{L}_I$ (see also Lin and Lee 1987).

At low temperatures, the system is ordered with spins on sublattice $\mathscr{L}'$ in a state which is

$$\begin{array}{lll} \text{ferromagnetic} & \text{if} & E_1 < E_2, E_3, E_4 \\ \text{antiferromagnetic} & \text{if} & E_2 < E_1, E_3, E_4 \\ \text{metamagnetic} & \text{if} & E_3 < E_1, E_2, E_4 \quad \text{or} \quad E_4 < E_1, E_2, E_3. \end{array} \tag{12}$$

Here,

$$\begin{aligned} -E_1 &= J+J'+|J_1+J_2+J_3+J_4| \qquad & -E_2 &= -J-J'+|J_1-J_2+J_3-J_4| \\ -E_3 &= -J+J'+|J_1-J_2-J_3+J_4| \qquad & -E_4 &= J-J'+|J_1+J_2-J_3-J_4|. \end{aligned} \tag{13}$$

As temperature rises, and depending on the relative strengths of the interactions J_r, one or more of the four equations in (11b) can be realised, signifying the occurrence(s) of phase changes. In particular a re-entrant transition occurs if any one equation admits two solutions, a possibility that has been shown (Vaks *et al* 1965) to occur in a limited region of the parameter space in the presence of competing interactions.

In summary, we have considered an Ising system with anisotropic and staggered interactions, and obtained its free energy and a sublattice spontaneous magnetisation in terms of those of the free fermion model. It should be pointed out that, as a free fermion model, the union jack Ising model is also equivalent to the chequerboard Ising model (Baxter 1986) and thus its partition function is factorisable into a product of four regular Ising model partition functions (Bazhanov and Stroganov 1985a, b).

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